

Do Empirical Pricing Factors Proxy for Innovations to State Variables in the ICAPM? A Direct Time-Series Test*

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Abstract

We propose a new time-series test of empirically motivated and empirically successful pricing factors within the intertemporal capital asset pricing model (ICAPM) framework. We discuss the relationship between the investment opportunity set, state variables, and pricing factors and show the importance of our test in understanding the nature of risk these factors represent. We also discuss the empirical difficulties in such tests and show how we get round these challenges. Lastly, we test if the risk premiums of size, value, momentum, and liquidity factors are associated with investors' concerns over future changes in market risk and find little evidence.

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1 Introduction

Since Fama and French (1993), empirical studies searching for new pricing factors have proliferated. Harvey et al. (2015) show that over 240 factors are reported in literature, not including those reported in unpublished working papers. This line of research contributed greatly to our understanding of the cross-section of equity returns. We have learned that various financial, macro, microstructure, behavioral and accounting variables are related to the level of expected return of a firm. We have more precise ways to measure the risks in an asset or fund. We also have numerous ideas for potentially very profitable trading strategies. Yet, we still try to understand the risks in those trading strategies and, ironically, we still try to figure out the nature of risk that the size and value factors represent.

Just like any other field in the social sciences, we rely on theories to understand the investment risks perceived by an investor and the corresponding compensation they expect. We need to build a model of investors' investment decisions and test it empirically to verify our understanding of the components that play an important part in the investment decision-making process and the implied relationship between risk and return. Along with the progress in empirical studies, theories have been developed to explain or interpret empirical findings. Interestingly enough, Merton's (1973) intertemporal capital asset pricing model (ICAPM) is still the most frequently cited theory to support a newly discovered empirical factor.¹ This is because the ICAPM can be easily linked to the arbitrage pricing theory (APT) of Ross (1976) and the linear factor pricing model (the framework in which empirical studies are done) and the ICAPM is silent on the identity of state variables (hence it imposes very loose restrictions on the pricing factors).

However, from early on in literature, Fama (1991) criticized the use of the ICAPM as a *fishing license* to support empirically motivated factors. Cochrane (2001) also argues that

¹The empirical factor, in this paper, refers to an empirically motivated pricing factor that is also empirically successful.

one can do much to make sure the candidate state variables really are plausible state variables for an explicitly stated optimization problem. While the static capital asset pricing model (CAPM) offers a well-defined and very intuitive investment opportunity set, the original form of ICAPM in Merton (1973) does not. The relationship between the investment opportunity set and the state variables that drives the changes of the investment opportunity set are so general that it is not easy to find meaningful restrictions for identifying the state variables.

In general, a good theory derives general implications with a small set of reasonable assumptions. However, it is sometimes beneficial to incorporate somewhat strong but reasonable assumptions to derive restrictive but more intuitive and testable implications. Campbell (1993) presents a discrete time version of the ICAPM in which the market return fully describes the investment opportunity set by assuming homoskedastic asset returns and consumption growth and adopting the log linearization of the budget constraint. Chen (2003), Sohn (2009), and Campbell et al. (2015) extend Campbell (1993) to accommodate time-varying volatilities of asset returns and show that the market volatility and the market return do represent the investment opportunity set. Nielsen and Vassalou (2002) and Brennan et al. (2004) model random changes in the instantaneous capital market line (ICML) and show that the investment opportunity set is fully described by the intercept (*i.e.*, risk-free rate) and the slope (*i.e.*, maximum Sharpe ratio) of the ICML.

In these more restrictive versions of the ICAPM, there are some aggregate variables such as the market return and the market volatility that represent the investment opportunity set, and they provide some testable implications on the state variable. Testing these implications against empirical factors not only verifies the theoretical consistency between the empirical factors and the ICAPM but also allows us to understand the nature of risk these empirical factors represent. We learn whether the risk premium of a certain empirical factor is due to investor concern over the future changes in the market return, the market volatility, or any other aggregate variable that represents the investment opportunity set.

Despite such benefits and importance, the time-series test of empirical factors has received less attention than it probably should because of some empirical difficulties. In the ICAPM, an empirical factor is supposed to be the innovations to the state variable that predicts one or more of the aggregate variables that represent the investment opportunity set; the empirical difficulty in the test arises from the fact that the state variable is *unknown* and *unobservable* in case of the empirical factors. In addition, the multiple aggregate variables describing the investment opportunity set further complicate the identification problem.

We propose a new time-series test for the empirical factors in the ICAPM framework. The new method resolves the empirical issues by directly linking the aggregate variable to the state variable that is reconstructed from empirical factors. We assume that the unknown and unobservable state variable follows an autoregressive process and invert the process to an infinite order moving-average process which is essentially a weighted sum of the empirical factors. Empirically, we adopt the mixed-data sampling (MIDAS) regression of Ghysels et al. (2002) and Ghysels et al. (2007) to implement the idea and reconstruct the state variable from the empirical factors. Then, we see if the reconstructed state variable predicts the aggregate variable. Our new methodology allows us to reconstruct the unknown state variable and understand the nature of risk in the empirical factor by directly linking the reconstructed state variable to our choice of aggregate variable for the investment opportunity set.

With the new test method, we investigate if the risk premiums of size, value, momentum, and liquidity factors are the compensation to investors' concern over the future market risk. Note that no such risk premium would exist if the market return was not predictable at all. The return predictability literature reports that the monthly market return is hard to predict with typically very low R^2 .² Welch and Goyal (2008) comprehensively reexamine the performance of well-known predictors of market returns and show that they have predicted

²We are interested in the monthly frequency of the market return because most of the empirical factors are studied with monthly data. In this case, according to Campbell's (1993) ICAPM, the factors are supposed to be the innovations to the monthly state variable that predicts the monthly market return.

poorly both in-sample and out-of-sample. This implies that innovation to the state variable, if it exists, would provide very noisy information about future changes in the market return. Accordingly, we find little evidence that any of the four factors have such information.

The papers that come close to this one, at least in part, are Petkova (2006) and Maio and Santa-Clara (2012). Petkova (2006) assumes the investment opportunity set is fully described by the excess market return and the yield curve. In time-series studies, Petkova (2006) essentially looks at the correlation between the innovations to the predictors for these variables and the Fama-French factors. In cross-sectional studies, which are her main focus, Petkova (2006) shows that the innovations to these state variables knock out the Fama-French factors. However, Petkova (2006) does not consider stock market volatility as one of the aggregate variables that describe the investment opportunity set. Chen (2003), Sohn (2009), and Campbell et al. (2015) theoretically show that stock market volatility should be included, and Ang et al. (2006), Adrian and Rosenberg (2008), and Sohn (2009) strongly support the idea empirically. When stock market volatility represents the investment opportunity set in the ICAPM, identifying the state variables becomes much more complicated, as we discuss later in this paper and in Maio and Santa-Clara (2012). This is because a good predictor for market return is likely to be a good predictor for market volatility, as shown theoretically by Mele (2007) and empirically by Christiansen et al. (2012). Under such circumstances, Petkova's (2006) approach cannot distinguish whether the Fama-French factors are priced because of their information about the future market return, future market volatility, or both.

Maio and Santa-Clara (2012) adopt a very rough measure for the state variable without any theoretical or econometric background: the empirical factor f is approximately the first difference of the state variable X , *i.e.*, $f_t \approx \Delta X_t = X_t - X_{t-1}$. If the empirical factors are innovations to the state variable as the ICAPM suggests, the specification implies that the state variable is a random walk with a unit root. Maio and Santa-Clara (2012) run a series of long-horizon predictive regressions in which the concern for the spurious regression arises

in this case. Also, the state variable specifications in Maio and Santa-Clara (2012) for the momentum and liquidity factors are exactly nested in our approach and roughly nested for the size and value factors. Our parameter estimates tell us a different story.

The next section discusses the relationship between the investment opportunity set and the aggregate variables that describe the investment opportunity set. Section 3 discusses the relationship between the state variable and pricing factors and presents the characteristics of a good potential state variable. Section 4 describes the empirical difficulties in understanding empirical factors in the ICAPM framework, which motivates the development of our new time-series test. Section 5 develops the new time-series test of the empirical factors within the ICAPM framework. Section 6 provides the empirical results using the market return as the aggregate variable. Finally, Section 7 concludes.

2 The ICAPM and Aggregate Measures for the Investment Opportunity Set

The key idea of the ICAPM is the recognition of the risks in the time-varying investment opportunity set available to investors. Originally in Merton (1973), the changes in the investment opportunity set are driven by a set of unidentified state variables. In response, investors would hold hedging portfolios to minimize the impact of the unexpected future changes in the investment opportunity set on their portfolios.

In the static CAPM, the investment opportunity set is constant, so it is easy to identify the boundary or frontier of the investment opportunity set relevant for investment decisions; the risk-free rate and the market risk premium fully describe the capital market line. On the other hand, Merton's (1973) ICAPM does not allow such a simple characterization of the investment opportunity set. The state variables X in Merton's (1973) ICAPM are defined as the common variables that drives the variations of the instantaneous mean returns and

volatilities of the instantaneous return processes for individual risky assets:

$$\frac{dP_i}{P_i} = \alpha_i(X, t)dt + \sigma_i(X, t)dz_i \quad (2.1)$$

$$dX = F(X)dt + G(X)dQ \quad (2.2)$$

where P_i is the price of a risky asset i which follows an Ito process with a Wiener process dz_i . The m -dimensional state variable vector X also follows an Ito process where F is a vector process $(f_1, f_2, \dots, f_m)^T$, G is a diagonal matrix with diagonal elements (g_1, g_2, \dots, g_m) , and dQ is the vector Wiener process $(dq_1, dq_2, \dots, dq_m)^T$. In Merton's (1973) fairly general setup, a sufficient set of statistics for the investment opportunity set at a given point in time is:

$$\{\alpha_i, \sigma_i, \rho_{ij}\} \quad (2.3)$$

where ρ_{ij} is the instantaneous correlation between the Wiener processes dz_i and dz_j . Unlike the case with the static CAPM, the investment opportunity set shown in Equation (2.3) cannot be fully described by a small set of aggregate variables such as the market risk premium or the risk-free rate. Thus, the restrictions that we can put on the state variables (which determine the investment opportunity set) for their identification must be pretty loose, which is why it is difficult to identify the state variables in Merton's (1973) ICAPM.

Naturally, stronger assumptions allow stronger and more intuitive implications. Campbell (1993) first came up with a meaningful restriction for identifying state variables. Assuming that variation in the consumption-wealth ratio is small and the conditional joint distribution of asset returns and consumption is homoskedastic, Campbell (1993) shows that the market return is an important aggregate variable that describes the investment opportunity set; investors try to hedge against unexpected changes in the future market return and news about the future market return becomes a legitimate pricing factor. Accordingly, the variables that are shown to forecast the market return should be chosen as

the state variables.

There are papers suggesting *multiple* aggregate variables describing the investment opportunity set. Chen (2003), Sohn (2009), and Campbell et al. (2015) extend Campbell (1993) to accommodate time-varying volatilities of asset returns and show that market volatility should also play an important role in describing the investment opportunity set relevant to investors' investment decisions. Nielsen and Vassalou (2002) and Brennan et al. (2004) model random changes in the ICML as driven by a number of state variables, which are independent Wiener processes, rather than directly model dynamics of asset returns. They further assume that the optimal portfolio on the ICML is a combination of the hedge portfolios and show that the investment opportunity set is fully described by the intercept (*i.e.*, risk-free rate) and the slope (*i.e.*, maximum Sharpe ratio) of the ICML.

In this paper, we use the term “*aggregate measure*” to refer to the aggregate variable, *e.g.*, the market return or the market volatility, that describes the investment opportunity set under a certain set of assumptions on a version of ICAPM. A different set of assumptions results in different implications on the aggregate measures that represent the investment opportunity set relevant to investors' decisions on the optimal portfolios. However, in all of these variants of the ICAPM, including the ICAPM of Merton (1973) itself, the asset returns are locally normal or lognormal and this makes investors mean-variance optimizers under fairly general conditions. Thus, it is not surprising that the aggregate measures describing the investment opportunity set are related to the capital market line of the static CAPM. These aggregate measures help us identify the state variables since the state variables are supposed to predict the future investment opportunity set described by the aggregate measures.

3 The State Variables and Pricing Factors

Regardless of the differences across models, the key results of the various versions of the ICAPM are driven by the time-varying investment opportunity set, which introduces additional systematic risks on the top of the market risk inherited from the static CAPM. The state variables are those forecasting these future changes in the investment opportunity set and, hence, the news on the future state variables carries information about the future changes in the investment opportunity set; the innovations to the state variables are the legitimate pricing factors within the ICAPM framework.

Sohn (2009) extends Campbell (1993) to allow heteroskedastic asset returns and provides a good example of a pricing equation that shows the relationship between the risk premium of an asset and innovations to the state variables:

$$E_t[r_{i,t+1} - r_{f,t+1}] + \frac{V_{ii,t}}{2} = \gamma V_{im,t} + (\gamma - 1)V_{ih,t} - \frac{(\gamma - 1)^2}{2(\sigma - 1)^2} V_{in,t} \quad (3.1)$$

where γ , σ , $r_{i,t+1}$ and $r_{f,t+1}$ are the coefficient of relative risk aversion, elasticity of intertemporal substitution, log return of asset i and risk-free asset, respectively. And, $V_{ii,t} = \text{Var}_t(r_{i,t+1})$, $V_{im,t} = \text{Cov}_t(r_{i,t+1}, r_{m,t+1})$, and, in particular:

$$V_{ih,t} = \text{Cov}_t\left(r_{i,t+1}, (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{m,t+1+j}\right) \quad (3.2)$$

and

$$V_{in,t} = \sigma^2 \text{Cov}_t\left(r_{i,t+1}, [E_{t+1} - E_t] \sum_{j=1}^{\infty} \rho^j \text{Var}_{t+j}[r_{m,t+j+1}]\right) \quad (3.3)$$

where ρ is a constant that comes from the log-linearization of the budget constraint. Equations (3.2) and (3.3) show that the market return and the market variance describe the investment opportunity set, and imply that there is a trading demand in investors'

perspectives to hedge out the unexpected changes in the investment opportunity set. The news on the future value of these variables is the pricing factor implied by the model in Sohn (2009). Ang et al. (2006) and Adrian and Rosenberg (2008) provide strong empirical supports for this idea by showing that the innovation to the market volatility is strongly and robustly priced across assets. The market volatility as an additional aggregate measure will be used as an example to show the complexity that the multiple aggregate measures would bring about in testing and understanding the empirical factors in the ICAPM framework.³

For an empirical implementation, Campbell (1993) suggests the Vector Autoregressive (VAR) factor model. Without loss of generality, the K -dimensional state vector process \mathbf{z}_t is assumed to follow the VAR process of the order one:

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \boldsymbol{\epsilon}_{t+1} \quad (3.4)$$

where the first two elements of \mathbf{z}_t are the market return and the estimate of market variance. Using the VAR model specified in Equation (3.4), we can rewrite part of Equations (3.2) and (3.3):

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{m,t+1+j} = \mathbf{e}'_1 \rho B (I - \rho B)^{-1} \boldsymbol{\epsilon}_{t+1} \quad (3.5)$$

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \hat{v}_{m,t+1+j} = \mathbf{e}'_2 \rho B (I - \rho B)^{-1} \boldsymbol{\epsilon}_{t+1} \quad (3.6)$$

where $\hat{v}_{m,t}$ is the conditional variance of the market return (*i.e.*, market variance estimate) at time t , and \mathbf{e}_1 is a vector with a first element of one and other elements are all zero. \mathbf{e}_2 is defined similarly. Following the terminology from Campbell (1993), the left-hand side of Equations (3.5) and (3.6) would be termed as the news on the market return and the market volatility, respectively.

For a more intuitive understanding of the relationship between the state variables and

³We will use the market variance and the market volatility interchangeably.

the pricing factors, we consider the following example processes for the demeaned state variables:⁴

$$r_{m,t+1} = b_d^{(m)} dy_t + \epsilon_{t+1}^{(m)} \quad (3.7a)$$

$$v_{m,t+1} = b_v^{(v)} v_{m,t} + b_p^{(v)} mp_t + \epsilon_{t+1}^{(v)} \quad (3.7b)$$

$$dy_{t+1} = b_d^{(d)} dy_t + \epsilon_{t+1}^{(d)} \quad (3.7c)$$

$$mp_{t+1} = b_p^{(p)} mp_t + \epsilon_{t+1}^{(p)} \quad (3.7d)$$

where $r_{m,t}$, $v_{m,t}$, dy_t , and mp_t are the market return, market variance, dividend yield and the growth rate in industrial production, respectively. For an easy and intuitive notation, $b_x^{(y)}$ denotes the coefficient for the predictor (regressor) x in the predictive regression for (regressand) y . The first two dependent variables, $r_{m,t}$ and $v_{m,t}$, describe the investment opportunity set, and the remaining two variables, dy_t and mp_t , are the state variables (*i.e.*, predictors) for $r_{m,t+1}$ and $v_{m,t+1}$, respectively. Both of these state variables follow $AR(1)$ processes. As is well known from literature on stock market volatility, market volatility is persistent and has its lagged market volatility as a predictor in addition to the growth rate in industrial production.

Then, from Equations (3.5) and (3.6), we obtain the equation for the news on the aggregate measures, market return and market volatility:

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{m,t+1+j} = \frac{\rho b_d^{(m)}}{1 - \rho b_d^{(d)}} \epsilon_{t+1}^{(d)} \quad (3.8)$$

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \hat{v}_{m,t+1+j} = \frac{\rho b_v^{(v)}}{1 - \rho b_v^{(v)}} \epsilon_{t+1}^{(v)} + \left[\frac{\rho^2 b_v^{(v)} b_p^{(v)}}{(1 - \rho b_v^{(v)})(1 - \rho b_p^{(p)})} + \frac{\rho b_p^{(v)}}{1 - \rho b_p^{(p)}} \right] \epsilon_{t+1}^{(p)} \quad (3.9)$$

⁴The state variables are demeaned for the simplicity and brevity of the theoretical arguments that follow. However, the same results hold even with the original variables.

and, the corresponding expected return-beta relationship would be

$$\begin{aligned}
E_t[r_{i,t+1} - r_{f,t+1}] + \frac{V_{ii,t}}{2} &= \gamma Cov_t(r_{i,t+1}, r_{m,t+1}) \\
&+ (\gamma - 1) \frac{\rho b_d^{(m)}}{1 - \rho b_d^{(d)}} Cov_t(r_{i,t+1}, \epsilon_{t+1}^{(d)}) \\
&- \frac{\sigma^2(\gamma - 1)^2}{2(\sigma - 1)^2} \frac{\rho b_v^{(v)}}{1 - \rho b_v^{(v)}} Cov_t(r_{i,t+1}, \epsilon_{t+1}^{(v)}) \\
&- \frac{\sigma^2(\gamma - 1)^2}{2(\sigma - 1)^2} \left[\frac{\rho^2 b_v^{(v)} b_p^{(v)}}{(1 - \rho b_v^{(v)})(1 - \rho b_p^{(p)})} + \frac{\rho b_p^{(v)}}{1 - \rho b_p^{(p)}} \right] Cov_t(r_{i,t+1}, \epsilon_{t+1}^{(p)})
\end{aligned} \tag{3.10}$$

where we can directly observe that $r_{m,t}$, $\epsilon_t^{(d)}$, $\epsilon_t^{(v)}$, and $\epsilon_t^{(p)}$ are the factors that determine the time-varying betas of the asset i . Together with the price of risks, they explain the risk premium for the asset i .

To begin with, Equation (3.10) verifies the basic idea of the ICAPM that the innovations to the predictors (or state variables) for the aggregate measures describing the investment opportunity set are the pricing factors. The innovation to the dividend yield is a systematic risk factor that represents the risk of unexpected changes in the market return. Both the innovations to the market variance and the growth rate in the industrial production are the systematic risk factors that represent the risk of unexpected changes in the market volatility.

Most importantly, Equations (3.8)-(3.9) give us hints about the characteristics of a good state variable in the ICAPM. First, news about the aggregate measures is mainly driven by the innovation to the *persistent* state variables, which is good because the empirical literature so far has found that most of the well-known predictors for market return or market volatility are persistent. It implies that the innovations to a persistent state variable would play an important role in pricing assets. To put it another way, among the many that predict aggregate measures, an important state variable for pricing assets would be the one with persistence. This comes from the fact that the denominators of all the coefficients for the innovations (*i.e.*, ϵ_{t+1} in Equations (3.8) and (3.9)) to the state variables are composed

of $(1 - \rho b_y^{(y)})$ where $b_y^{(y)}$ is the coefficient of the $AR(1)$ state variable process. Note that:

$$b_y^{(y)} = \frac{Cov(y_{t+1}, y_t)}{Var(y)} = Corr(y_{t+1}, y_t) = acf(y, 1) < 1 \quad (3.11)$$

where $acf(y, 1)$ denotes the first-order autocorrelation function of y .⁵ A persistent process of $AR(1)$ has a high serial correlation, which would make $(1 - \rho b_y^{(y)})$ very small; this would also make the news on an aggregate measure very sensitive to innovation to the persistent state variable.

Second, innovation to the state variable with *good forecasting power* on the aggregate measure for the investment opportunity set is an important pricing factor. This is because the numerators of the coefficients for innovations to state variables in Equations (3.8) and (3.9) are determined by the coefficients for the state variables in the predictive regressions in Equations (3.7a) and (3.7b). The coefficients in the predictive regressions, for example in Equation (3.7a), can be seen as:

$$b_d^{(m)} = \frac{Cov(r_{m,t+1}, dy_t)}{Var(dy_t)} = Corr(r_{m,t+1}, dy_t) \frac{STD(r_{m,t+1})}{STD(dy_t)} \quad (3.12)$$

High correlation in Equation (3.12) (*i.e.*, good forecasting power) leads to large coefficients in predictive regressions and large coefficients in Equation (3.8).

In the ICAPM, the unexpected future changes in the investment opportunity set are the key systematic risks, and state variables drive these changes. Thus, innovations to the state variables carry information about future changes in the investment opportunity set and become the legitimate pricing factors under the ICAPM framework. In light of this, an ideal state variable in the ICAPM would have the following properties: (*i*) [Predictability] basically, a good state variable is a good predictor for an aggregate measure of investment opportunity set, in which case innovation to the state variable would have much information

⁵We assume that $v_{m,t}$ and mp_t are uncorrelated for the sake of brevity of the theoretical arguments.

about future unexpected changes in the aggregate measure; and (ii) [Persistence] persistence makes a state variable a good one, *ceteris paribus*, because it makes the state variable more predictable, so its innovation contains more precise information about future changes in the aggregate measure. Accordingly, innovations to the state variables with the above properties make strong legitimate pricing factors in the ICAPM.

Aside from the characteristics of a good state variable, several other interesting points can be learned from this simple exercise. First, the sign of correlation between news on the aggregate measure for the investment opportunity set and the innovations to the corresponding state variable is determined by the sign of the correlation between the aggregate measure and the lagged state variable itself. In other words, the signs of the coefficients for the innovation $\epsilon_{t+1}^{(s)}$ in Equations (3.8) and (3.9) are solely determined by $b_s^{(A)}$ where ‘A’ stands for an aggregate measure and ‘s’ the corresponding state variable. This sheds some light on the information about news on the aggregate measure embedded in the state variable innovation. For example, suppose the dividend yield positively predicts the market return in Equation (3.7a); *i.e.*, $b_d^{(m)} > 0$. Then, the innovation to the dividend yield also positively correlates with the news on the market return in Equation (3.8), which implies that unexpected increase in the dividend yield would be likely to accompany the increase in market return. The sign relationship is due to the observation that the sign of $(1 - \rho b_y^{(y)})$ is positive and hence the sign of coefficients for the innovations to the state variables in Equations (3.8) and (3.9) are solely determined by the sign of the numerators.⁶

Second, the sign of the price of risk for each of the innovations to the state variables in Equation (3.10) is determined by the sign of the coefficient from the corresponding predictive regression between the aggregate measure and state variable.⁷ For all the risk premium terms related to the hedging demand in Equation (3.10), the sign of the price of risk is solely

⁶ ρ is not a discount factor but a coefficient from the loglinearization of the budget constraint. However, it is known to be numerically similar to a discount factor; the value of ρ is close to but less than one.

⁷We assume that the reasonable value for the coefficient of relative risk aversion (γ) is larger than 1. Numerous studies assume that the reasonable value of γ is around 7 and report that the empirical estimate is a lot higher.

determined by $b_{(s)}^{(A)}$ where ‘ A ’ stands for an aggregate measure and ‘ s ’ a corresponding state variable because the rest of the components for the price of risk are all positive.⁸ In Equation (3.10), if the dividend yield positively predicts the market return, the price of risk for the innovation to the dividend yield would be also positive. On the other hand, if industrial production growth positively predicts market volatility, the corresponding price of risk is negative. Interestingly, if a state variable positively predicts both the market return and market volatility, innovation to the state variable would command a positive risk premium for the future market risk and a negative risk premium (or risk discount) for future market uncertainty; the innovation to such a state variable can have a positive, negative, or nil price of risk depending on the size of the price of risk on the future market risk and the future market uncertainty.

Lastly, the state variable dynamics in Equations (3.7a)-(3.7d) and the pricing equation in Equation (3.10) help us understand why the literature on cross-section of equity returns reports so many empirical factors, as is pointed out in Harvey, Liu, and Zhu (2015). The market volatility process in Equation (3.7b) has two predictors, $v_{m,t}$ and mp_t , and the pricing equation in Equation (3.10) shows that each of the innovations to these two predictors commands separate risk premiums even though both innovations are priced for the same reason: they have information about the future changes in the market volatility. This implies, in general, all the innovations will be separately priced if each of the state variables, from which the innovations come from, has independent predictive power over an aggregate measure of the investment opportunity set. To put it differently, an empirical finding that some two factors survive in cross-sectional regression together does not imply that these two factors represent two different systematic risks. It can be that both factors represent the same kind of risk, say the future market uncertainty, but have independently useful information on future changes in market volatility.

⁸ $b_{(v)}^{(v)}$ is positive because the market volatility is persistent.

4 Understanding Empirical Pricing Factors

As shown in Harvey et al. (2015), there are more than 240 factors reported in the literature on the cross-section of equity returns. Most of them are empirically motivated and, not surprisingly, the most empirically successful factors are this kind. Many of these papers cite Merton's (1973) ICAPM to provide a theoretical foundation without providing specific description of the investment opportunity set or state variables. From early on in the literature, Fama (1991) criticized the use of the ICAPM as a *fishing license* to support the empirically motivated factors.

The loose link between an empirically motivated factor and the ICAPM is a problem partly because the state variable in the ICAPM is not really free and there are restrictions on the state variable imposed by the ICAPM, but most importantly because it does not allow us to understand what systematic risk the empirically motivated factor represents. Cochrane (2001) argues that one could do a lot to insist that the factor-mimicking portfolios are the projections of some identifiable state variables on the space of returns, and much can be done to make sure the candidate state variables really are plausible state variables for an explicitly stated optimization problem.

Numerous papers try to provide risk-based explanations for empirically successful yet empirically motivated factors such size, value, and momentum. The first line of this research naturally focused on the relationship between the factors and important macroeconomic or financial variables such as default premium, term spread, GDP growth and industrial production growth. The papers do not provide further theoretical restrictions on the state variables than Merton's (1973) ICAPM and, hence, implicitly assume the time variations of the sufficient statistics in Equation (2.3) are driven by their choice of macroeconomic or financial variables (the state variables in the context of ICAPM). Since the dependency relations are at the individual stock level, it is hard to show if these state variables determine the return dynamics of every individual asset in the market, and these papers are typically

silent on this. More importantly, if the empirically successful factors do proxy the innovation to these macroeconomic or financial variables, then the innovations to these macroeconomic or financial variables themselves should be priced across assets, which brings us back to Chen et al. (1986). However, as shown by Shanken and Weinstein (2006), most of these are not robust or not priced at all except for industrial production growth.

Campbell (1993) shows that the market return sufficiently summarizes the investment opportunity set necessary for the investment decision under the assumption of homoskedastic asset returns. In Campbell's (1993) discrete time version of the ICAPM, the state variables should be chosen such that they predict the market return. Thus, the empirically successful factors should proxy the innovations to the predictors of the market return. To cope with the critiques from Fama (1991) and Cochrane (2001), it is very important to look into the time-series relations between the investment opportunity set and the chosen state variables. It is important because they are the ICAPM's restrictions on the state variables and because they are the sources of concern for investors; the market return as the relevant summary statistics for the investment opportunity comes from individual portfolio optimization problem exactly as the recommendation from Cochrane (2001).

Understanding the empirical factors within the ICAPM framework is not a straightforward or easy task. First of all, according to Campbell (1993), these empirical factors are supposed to be the innovations to some *unknown* and *unobservable* state variable that predicts the market return. It is very difficult to show empirically that the empirical factors proxy the innovations to a variable that is unknown and predicts the market return.

Second, another empirical difficulty in understanding the empirical factors in Campbell's (1993) ICAPM framework is that the market return is not very predictable. Welch and Goyal (2008) reexamine the performance of variables that have been well known to predict the market return and find that they have predicted poorly both in-sample and out-of-sample. As is shown in Equations (3.5) and (3.8), the innovations to the predictor of the

market return should contain information about the news on the market return. However, considering the typical low R^2 reported in return predictability literature, it is very likely that innovations to the state variable carry very noisy information about news on the market return, and it would be even harder to link the empirical factors to these noisy innovations.

Third, *multiple* aggregate measures that describe the investment opportunity set further complicate the empirical difficulties in understanding the nature of risk the empirical factors represent. It is almost natural that the variance of the market return becomes another aggregate measure when heteroskedastic asset returns are allowed in Campbell's (1993) ICAPM as shown in Chen (2003), Sohn (2009) and Campbell et al. (2015). The addition further complicates our attempt to understand the empirical factors especially when the market return and market volatility share the same predictors. This is indeed true as shown theoretically by Mele (2007) and empirically by Christiansen et al. (2012). To incorporate the idea and analyze its effect on the pricing factors, we can change the VAR state variable dynamics in Equations (3.7a)-(3.7d) as follows:

$$r_{m,t+1} = b_c^{(m)} c_t + \epsilon_{t+1}^{(m)} \quad (4.1a)$$

$$v_{m,t+1} = b_v^{(v)} v_{m,t} + b_c^{(v)} c_t + \epsilon_{t+1}^{(v)} \quad (4.1b)$$

$$c_{t+1} = b_c^{(c)} c_t + \epsilon_{t+1}^{(c)} \quad (4.1c)$$

where c_t is the common predictor for the market return and market volatility. Then, the corresponding pricing factors are:

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{m,t+1+j} = \frac{\rho b_c^{(m)}}{1 - \rho b_c^{(c)}} \epsilon_{t+1}^{(c)} \quad (4.2)$$

$$(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \hat{v}_{m,t+1+j} = \frac{\rho b_v^{(v)}}{1 - \rho b_v^{(v)}} \epsilon_{t+1}^{(v)} + \left[\frac{\rho^2 b_v^{(v)} b_c^{(v)}}{(1 - \rho b_v^{(v)})(1 - \rho b_c^{(c)})} + \frac{\rho b_c^{(v)}}{1 - \rho b_c^{(c)}} \right] \epsilon_{t+1}^{(c)} \quad (4.3)$$

It should be noted that the news on both the market return and market volatility is

correlated with the innovations to the common predictor c_t . This implies that innovations to the common predictor c_t have information about news on both aggregate measures. Consequently, the fact that a good market return predictor is highly likely to also be a good market volatility predictor makes it even harder to identify the nature of risk the empirical factors represent. Suppose an empirical factor proxies the innovations to the state variable that predicts the market volatility. The factor would correlate with the innovations to the common predictor c_t since the innovations contain news on both the market return and market volatility. However, the correlation does not imply that the factor proxies the news on the market return. Hence, for an empirical factor that is correlated with the innovation to the common predictor, we cannot tell if the empirical factor has information about the future changes in the market return or the market volatility.

Fourth, the cross-sectional regressions are not very helpful in distinguishing the two sources of risk because innovations to the common predictor c_t would also command a premium on both systematic risks as follows:

$$\begin{aligned}
& E_t[r_{i,t+1} - r_{f,t+1}] + \frac{V_{ii,t}}{2} \tag{4.4} \\
&= \gamma \text{Cov}_t(r_{i,t+1}, r_{m,t+1}) \\
&\quad - \frac{\sigma^2(\gamma - 1)^2}{2(\sigma - 1)^2} \frac{\rho b_v^{(v)}}{1 - \rho b_v^{(v)}} \text{Cov}_t(r_{i,t+1}, \epsilon_{t+1}^{(v)}) \\
&\quad + \left[\underbrace{(\gamma - 1) \frac{\rho b_c^{(m)}}{1 - \rho b_c^{(c)}}}_{\text{price of future market risk}} - \underbrace{\frac{\sigma^2(\gamma - 1)^2}{2(\sigma - 1)^2} \left[\frac{\rho^2 b_v^{(v)} b_c^{(v)}}{(1 - \rho b_v^{(v)})(1 - \rho b_c^{(c)})} + \frac{\rho b_c^{(v)}}{1 - \rho b_c^{(c)}} \right]}_{\text{price of future market volatility risk}} \right] \text{Cov}_t(r_{i,t+1}, \epsilon_{t+1}^{(c)})
\end{aligned}$$

As briefly discussed in Section 3, the sign of the price of risk on the innovation to the common predictor c_t can be positive, negative, or even zero. The fact that the innovation (factor) to the common predictor, *e.g.*, dividend yield, knocks out an empirical factor in the cross-sectional regression does not allow us to say whether the empirical factor has the pricing information subsumed in the news for the market return or market volatility

unless we examine the signs of $b_c^{(m)}$ and $b_c^{(v)}$ at the least. Suppose an empirical factor has a positive price of risk. The positive price of risk in Equations (3.10) and (4.4) implies three possibilities: the *unknown* state variable X , to which the empirical factor is the innovation (1) predicts the market return only and $b_X^{(m)} > 0$; (2) predicts the market volatility only and $b_X^{(v)} < 0$; or (3) predicts both and (i) $b_X^{(m)} > 0, b_X^{(v)} < 0$, or (ii) $b_X^{(m)} > 0, b_X^{(v)} > 0$ and:

$$(\gamma - 1) \frac{\rho b_X^{(m)}}{1 - \rho b_X^{(X)}} > \frac{\sigma^2(\gamma - 1)^2}{2(\sigma - 1)^2} \left[\frac{\rho^2 b_v^{(v)} b_X^{(v)}}{(1 - \rho b_v^{(v)})(1 - \rho b_X^{(X)})} + \frac{\rho b_X^{(v)}}{1 - \rho b_X^{(X)}} \right] \quad (4.5)$$

Since the innovation to the common predictor contains news on both the future market return and the future market volatility and also commands the risk premiums for both, we cannot distinguish the above three cases even if the innovation knocks out the empirical factor in the cross-sectional regression. Examining the signs of $b_c^{(m)}$ and $b_c^{(v)}$ to see if they are consistent with the cases above might be helpful in distinguishing the three cases. However, if it turns out $b_c^{(m)} > 0, b_c^{(v)} < 0$, then we cannot tell the difference between the three because the signs imply the prices of risk for both the future market return and the future market volatility are positive. Hence, it is best to reconstruct the unknown state variable itself to which the empirical factor is the innovation and check the statistical significance of $b_X^{(m)}$ and $b_X^{(v)}$. This again shows the importance of the time-series test in understanding the nature of risk the empirical factor represents.

Despite its importance, the time-series test on the empirical factors to discover their relation to the state variable, which is subject to restrictions from the ICAPM has received less attention than it should because of the various empirical difficulties discussed so far. A few good papers actually try to address this in their time-series studies to check if the empirical factors meet the restrictions. Petkova (2006) and In and Kim (2007) basically investigate the correlation between the empirically successful factors (SMB and HML) and the innovations to a set of well-known predictors of the market return. However, the examination of the correlation alone will not get us far because of the issues we have discussed

in this section. Petkova (2006) also carries out cross-sectional studies but, for the same reason discussed previously, the results do not really allow us to distinguish the two sources of systematic risk for the empirical factors in the presence of the market volatility as an aggregate measure.

Maio and Santa-Clara (2012) try to reconstruct the original predictor (or the state variable) by summing up the factors. The reconstructed state variable would be similar to the cumulative return on the zero-cost portfolio following the strategies of SML and HML.⁹ However, the simple aggregation of some innovation series would be a random walk, which would also have a unit root. The long-horizon predictive regression with such a variable as a regressor would likely lead to a spurious regression, which would further complicate the empirical test.

With their estimates of the state variables X_t , Maio and Santa-Clara (2012) examine the consistency within the ICAPM between the sign of the coefficient from the predictive regression ($b_X^{(m)}$ and $b_X^{(v)}$) and the sign of the corresponding price of risk. However, they discuss it separately for the two different aggregate measures of the market return and market volatility. Since they derive their sign implications from the ICAPM, relying on just intuition but without any rigorous theoretical background and do it separately for the two aggregate measures, their framework does not show the complexity in the signs in the presence of the common predictor as we discuss here.

5 Direct Time-Series Test

If there is a single aggregate measure for the investment opportunity set, it would be easier (relatively) to investigate if an empirical pricing factor could be understood in the ICAPM

⁹Maio and Santa-Clara (2012) obtain the state variable X_t for liquidity and momentum factors by summing of the corresponding factors over 60 months: $X_t = \sum_{s=t-59}^t f_s$. They use a different approach for the size and value factors, but, essentially the approach is the same as the previous one in the sense that $SMB_t \simeq \Delta X_t(SMB)$ and $HML_t \simeq \Delta X_t(HML)$.

framework. However, if there are multiple aggregate measures and some state variables are common predictors for some of them, it is not so clear empirically how to relate the empirical factor to the aggregate measure in the investors' optimization problem as was suggested by Cochrane (2001). The ICAPMs of Chen (2003), Sohn (2009) and Campbell et al. (2015) are typical examples of this, where the market volatility makes another aggregate measure.

We will develop an empirical methodology that allows us to directly link an empirical factor to a specific aggregate measure for the investment opportunity set. Recall the discussion from the previous section that the empirical difficulty of our task comes from the fact that the empirical factor of interest is an innovation to an unknown predictor (or state variable) of an aggregate measure. We reconstruct the *unknown* state variable from the empirical factor with an assumption that the state variable follows a certain stochastic process. We first assume that the state variable follows an AR(1) process and then try to accommodate more general processes.

Our assumption of the process of the state variable is consistent with our discussion of the state variable in Section 3 and backed by numerous papers in return predictability literature. All the market return predictors examined in Welch and Goyal (2008) are persistent and well described by the AR(1) process. Accordingly, they determine the statistical significance of the predictability by F -statistics from bootstrapping which assumes an AR(1) process for all the predictors. Conrad and Kaul (1988) investigate the stochastic nature of expected returns and find the variation through time in expected returns is well characterized by an AR(1) process. Kojien and van Binsbergen (2010) propose a latent variable approach within a present-value model to estimate the time series of expected returns and expected dividend growth rates of the aggregate stock market. They model both expected returns and expected dividend growth rates as an AR(1) process, and find that these are good predictors of realized returns and realized dividend growth rates.

To reconstruct the state variable from its innovations, we use the moving average

representation of AR(1) process. When the state variable process that is supposed to follow AR(1) is converted into the moving average process, the state variable can be represented with its innovation which is the corresponding pricing factor. Then, the predictive regression for the aggregate measure on the moving average representation of the state variable with the empirical factor is estimated with the MIDAS regression of Ghysels et al. (2002) and Ghysels et al. (2007) to see the statistical significance of the prediction relation.

5.1 MIDAS Regression

A MIDAS regression model generalizes a distributed lag model to accommodate time-series data sampled at different frequencies. For the scope of this paper, it is sufficient to adopt a distributed lag model to estimate and reconstruct the state variable from empirical factors. However, we would like to stay open to further extension that incorporates higher frequency data even though we do not cover it here. Also, a technically similar application of the MIDAS regression is successfully used in investigating the risk-return tradeoff in Ghysels et al. (2005) and the volatility forecasting in Ghysels et al. (2006).

The MIDAS regressions are essentially tightly parameterized, reduced-form regressions that involve processes sampled at different frequencies. A simple linear MIDAS regression takes the form of:

$$Y_{t+1} = b_0 + b_1 B(L^{1/m}) Z_t^{(m)} + \varepsilon_{t+1}^{(m)} \quad (5.1)$$

where $B(L^{1/m}) = \sum_{j=0}^J B(j) L^{j/m}$ is a polynomial of length J in the $L^{1/m}$ operator, and $L^{j/m} Z_t = Z_{t-j/m}$. m represents the frequency of data sampling for Z . Suppose that Y_t is sampled at some fixed, say monthly, sampling frequency. Then, $Z^{(m)}$ is sampled m times faster, so $Z^{(22)}$ is sampled daily when the average number of trading days in a month is 22.¹⁰

Two most important decisions that should be made to implement the MIDAS regression

¹⁰For our current purposes in this paper, we can fix $m = 1$, which leads to a typical distributed lag model; the data frequency is fixed at a month.

model in Equation (5.1) are the order and the structure of the polynomial $B(L^{1/m})$.

As will be further discussed in the next subsection, we will apply the MIDAS regression model to an infinite order moving average process, which implies, in theory, the length of the polynomial, J , is infinite. However, we also know from the stationarity condition that the coefficients for the series of shocks in the infinite order moving average process should eventually be time-decaying. Thus, we may set J a finite number and ignore the shocks that come after that lag.

Our choice of the structure for the lag polynomial is based on its flexible ability to accommodate various inverted infinite order MA processes. We choose the exponential Almon lag polynomial introduced in Ghysels et al. (2007):

$$B(L; \theta) = \sum_{j=0}^J \frac{\exp(\theta_1 j + \dots + \theta_K j^K)}{S_e} L^j \quad (5.2)$$

where

$$S_e = \sum_{i=0}^J \exp(\theta_1 i + \dots + \theta_K i^K) \quad (5.3)$$

Note that the coefficients are normalized such that the sum of them will be one. For our current setup, it is good enough to set $m = 1$.¹¹ The choice of K determines the flexibility of the lag structure (or weighting scheme) that can be represented by Equation (5.2). As you increase K , you have more terms and parameters in the exponential function. The lag polynomial with lower K is nested in the one with the higher K . The simplest form with $K = 1$ has exponentially time-decaying weights and will be used in Section 5.2. To deal with more general state variable processes, Section 5.3 will adopt the weighting scheme with $K = 2$. Figure 1 shows the various lag weighting structures the exponential Almon lag polynomial can take with different sets of parameters. The parameter choices of (1) and (3) in the figure have increasing weights and, hence, are not appropriate for stationary infinite

¹¹Our choice of the frequency of data sampling (*i.e.*, $m=1$) implies that we have the same data sampling frequency on both sides of Equation (5.1): monthly data.

order MA processes. The shapes of weighting structure in (2) and (4) in Figure 1 seem appropriate for our purpose; the former has a single hump-shaped weighting scheme and the latter has monotonically decreasing weights.¹²

5.2 The State Variable Follows an AR(1)

In the ICAPM framework, the empirical factors are supposed to be the innovations to the state variables that predict some aggregate measures that describe the relevant investment opportunity set as shown in Equations (3.7a)-(3.9) and Equations (4.1a)-(4.3). To apply the empirical methodology developed here, the investment opportunity set related aggregate measures need not be the ones discussed in Section 3 or Section 4. Depending on a set of assumptions, a version of the ICAPM has a different set of the aggregate measures as in Nielsen and Vassalou (2002) and Brennan, Wang, and Xia (2004). As long as there is an aggregate measure and a set of state variables is supposed to predict the aggregate measure, we can use the method to see if an empirical factor represents the systematic risk related to the aggregate measure that comes from investors' optimization problems.

Let A_t and X_t be an aggregate measure for the investment opportunity set at time t and a state variable at time t that has the predictive power for the aggregate measure, respectively, as follows:

$$A_{t+1} = a_0 + a_1 X_t + \epsilon_{t+1}^A \quad (5.4)$$

with the assumption that the state variable follows an AR(1) process:

$$X_{t+1} = \phi X_t + \epsilon_{t+1}^X \quad (5.5)$$

where X_t is demeaned for simplicity of the argument. Under some regularity conditions, an

¹²The signs of θ_1 and θ_2 do not completely determine the shape of a weighting scheme. For the complete description of the weighting scheme, one needs to take into account the quadratic functional features of the weighting scheme and the finite number of lags.

AR(1) can always be expressed as an MA(∞):

$$X_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}^X \quad (5.6)$$

Then, we can substitute this back into Equation (5.4) to obtain:

$$A_{t+1} = a_0 + a_1 \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}^X + \epsilon_{t+1}^A \quad (5.7)$$

Note that ϵ^X is the innovation to the state variable X and is supposed to be the pricing factors, f^X , related to the aggregate measure A . For a more intuitive representation, we replace ϵ^X with f^X :

$$A_{t+1} = a_0 + a_1 \sum_{i=0}^{\infty} \phi^i f_{t-i}^X + \epsilon_{t+1}^A \quad (5.8)$$

The parameters a_0 , a_1 and ϕ can be estimated with the MIDAS regression as specified in Equations (5.1) and (5.2) with $K = 1$ when:

$$a_0 = b_0, \quad a_1 = \frac{b_1}{S_e}, \quad \phi = e^{\theta_1} \quad (5.9)$$

Hence, for a given aggregate measure A and an empirical factor f^X , the model in Equation (5.8) can be estimated and we can see if the parameter a_1 is statistically significant to determine whether the empirical factor represents the systematic risk related to the aggregate measure. It should be noted that the application of the MIDAS regression model implies that the estimated ϕ is restricted to be positive (see Equation (5.9)). Note that ϕ is not only the weight of the empirical factors as in Equation (5.8) but also the coefficient for the AR(1) process for the state variable as in Equation (5.5). Recall our discussion on the characteristics of an ideal state variable in Section 3. One of them was persistence and the positivity of ϕ is consistent with this property.

5.3 More General State Variable Processes

The structure of the lag polynomial we adopted is quite flexible and the MIDAS regression of innovations can represent various infinite order MA processes exactly and sometimes approximately. One of these processes of interest to us is the infinite order MA representation of the AR(2) process. The analysis with the state variable process following AR(2) itself is interesting, but also extends our analysis in Section 5.2 in the sense that it addresses the issues with autocorrelated innovations (factors).

The way the ICAPM interprets the empirical factors is that they are innovations to some state variables, and, as in Equation (5.8), they are treated as innovations with no autocorrelation. However, in practice, some empirical factors show small but statistically significant serial correlation.

Consider a case in which a state variable follows AR(1) but the innovations are autocorrelated:

$$X_{t+1} = \phi X_t + \epsilon_{t+1}^X \quad (5.10a)$$

$$\epsilon_{t+1}^X = \delta \epsilon_t^X + u_{t+1} \quad (5.10b)$$

where u_t is white noise. Then, it can be easily shown that:

$$X_{t+1} = (\phi + \delta)X_t - \delta\phi X_{t-1} + u_{t+1} \quad (5.11)$$

which implies that the state variable X is indeed an AR(2) process.¹³ In our application, we can estimate the innovations in the empirical factors by taking out the autocorrelated component as in Equation (5.10b) and reconstruct the state variables from these innovation series assuming the process in Equation (5.11).

¹³Note that $\epsilon_t^X = X_t - \phi X_{t-1}$ from Equation (5.10a) and we can plug this into Equation (5.10b) to obtain $\epsilon_{t+1}^X = \delta(X_t - \phi X_{t-1}) + u_{t+1}$. Plug this back into Equation (5.10a) to get Equation (5.11).

To understand the structure of coefficients for the infinite order MA representation of a *general* AR(2) process, it helps to factorize the lag polynomial. Assuming a state variable follows a *general* AR(2) process, we have:

$$X_{t+1} = \phi_1 X_t + \phi_2 X_{t-1} + \epsilon_{t+1}^X \quad (5.12)$$

which is equivalent to:

$$(1 - \phi_1 L - \phi_2 L^2) X_{t+1} = \epsilon_{t+1}^X \quad (5.13)$$

We can factorize the lag polynomial and obtain:

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \quad (5.14)$$

which implies:

$$\phi_1 = \lambda_1 + \lambda_2 \quad (5.15)$$

$$\phi_2 = -\lambda_1 \lambda_2 \quad (5.16)$$

Using the factorized lag polynomial, we can rewrite Equation (5.13) with

$$X_{t+1} = \frac{\epsilon_{t+1}^X}{(1 - \lambda_1 L)(1 - \lambda_2 L)} \quad (5.17)$$

Then, using the partial fraction representation and inverting $(1 - \lambda L)$, we obtain the following

infinite order MA representation for the state variable process:

$$X_t = \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \frac{1}{1 - \lambda_1 L} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \frac{1}{1 - \lambda_2 L} \right] \epsilon_t^X \quad (5.18)$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} \lambda_1^j \epsilon_{t-j}^X + \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{j=0}^{\infty} \lambda_2^j \epsilon_{t-j}^X \quad (5.19)$$

$$= \sum_{j=0}^{\infty} \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_1^j + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_2^j \right] \epsilon_{t-j}^X \quad (5.20)$$

$$= \sum_{j=0}^{\infty} \left[\frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} \right] \epsilon_{t-j}^X \quad (5.21)$$

Note that the response of X to a shock ϵ is a sum of two exponentials with two symmetric parameters λ_1 and λ_2 in Equation (5.20), *i.e.*, the lag weight for a shock will be the same even if we switch λ_1 and λ_2 . Under the stationarity condition, *i.e.*, $|\lambda_i| < 1$ for $i = 1, 2$, the lag polynomial in Equation (5.21) takes various forms, including oscillating ones. Suppose an empirical factor f^X proxies the innovations to an AR(2) state variable X that predicts an aggregate measure A . Then, it can be written as:

$$A_{t+1} = a_0 + a_1 \sum_{j=0}^{\infty} \left[\frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} \right] f_{t-j}^X + \epsilon_{t+1}^A \quad (5.22)$$

Regarding the state variable process with autocorrelated errors, we can rewrite Equation (5.11) with lag polynomials as follows:

$$(1 - (\phi + \delta)L + \delta\phi L^2)X_{t+1} = (1 - \phi L)(1 - \delta L)X_{t+1} = u_{t+1} \quad (5.23)$$

Thus, in our setup in Equations (5.10a)-(5.10b), the state variable process has the following infinite order MA representation:

$$X_t = \sum_{j=0}^{\infty} \left[\frac{\phi^{j+1} - \delta^{j+1}}{\phi - \delta} \right] u_{t-j} \quad (5.24)$$

Note that the lag weighting structure for u_t in Equation (5.24) is exactly same as the one for ϵ_t^X in Equation (5.21). As discussed in Section 5.2, it is reasonable to assume that ϕ is positive and close to one. The sign of δ can be different depending on the choice of empirical factor and the data frequency. It turns out that all the (monthly) empirical factors examined in this paper have *positive* but small δ regardless of their statistical significance. With the stationarity condition, we can limit the ranges of ϕ and δ to be:

$$0 \ll \phi < 1, \quad 0 < \delta \ll 1 \quad (5.25)$$

For the above ranges of values, all the coefficients to the lag polynomial in Equation (5.24) are non-negative because $\phi^{j+1} > \delta^{j+1}$ whenever $\phi > \delta$ for all $j \geq 0$. The lag weighting scheme given in Equation (5.24) with the parameter values in the range in Equation (5.25) is either monotonically decreasing or single-hump shaped.¹⁴ This lag structure is well approximated by the exponential Almon lag polynomial with $K = 2$ because it also represents either monotonically decreasing or single-hump shaped weighting scheme as we discussed in Section 5.1. Thus, as in Section 5.2, we look into the following predictive relation:

$$A_{t+1} = a_0 + a_1 X_t + \epsilon_{t+1}^A \quad (5.26)$$

$$= a_0 + a_1 S_\lambda \sum_{j=0}^{\infty} \frac{1}{S_\lambda} \left[\frac{\phi^{j+1} - \delta^{j+1}}{\phi - \delta} \right] u_{t-j} + \epsilon_{t+1}^A \quad (5.27)$$

$$\approx a_0 + a_1 S_\lambda \sum_{j=0}^J \frac{\exp(\theta_1 j + \theta_2 j^2)}{S_e} u_{t-j} + \epsilon_{t+1}^A \quad (5.28)$$

where

$$S_\lambda = \sum_{j=0}^{\infty} \left[\frac{\phi^{j+1} - \delta^{j+1}}{\phi - \delta} \right] u_{t-j} \quad (5.29)$$

Note that lag weights are normalized in Equation (5.27) to make it comparable to the MIDAS

¹⁴ $f(x) = (\lambda_1^x - \lambda_2^x)/(\lambda_1 - \lambda_2)$ is basically positive and single-hump-shaped for $x > 0$ and $0 < \lambda_1, \lambda_2 < 1$. The $f(x)$ has its max at $x^* = \ln(\ln \lambda_2 / \ln \lambda_1) / \ln(\lambda_1 / \lambda_2)$. Hence, if $x^* < 1$, then lag weights will be monotonically decreasing.

regression model in Equation (5.28). The MIDAS regression model in Equation (5.28) does not allow us to separately estimate a_1 and S_λ , but it is sufficient to see the statistical significance of $b_1 = a_1 S_\lambda$ to determine whether the state variable predicts the aggregate measure of the investment opportunity set.

To find out how close the approximation in Equation (5.28) is for Equation (5.27), we try to match the normalized lag weights in Equation (5.27) for various sets of parameter values (ϕ and δ , or λ_1 and λ_2 in general) with the exponential Almon lag weighting scheme in Equation (5.28). The parameter values θ_1 and θ_2 are chosen to minimize the mean squared error between the two lag structures. Figure 2 shows the results of these exercises.

Panel A presents the most probable case under the setup in Equations (5.10a)-(5.10b). The state variable is very persistent ($\phi = 0.95$) as we already discussed in Section 3 and its persistence is empirically consistent with most of well-known market return predictors. The δ value chosen ($\delta = 0.15$) is also based on the empirical values from the empirical factors. Panel A of Figure 2 confirms that the approximation is very good and it supports our approximation in Equation (5.28) under the assumed processes in Equations (5.10a)-(5.10b).

The rest of the panels in Figure 2 show the goodness of fit in the lag structure approximations when the state variable follows a *general* AR(2) process. For these, we assume that the coefficient to the first lagged state variable in Equations (5.12) and (5.15) is positive:

$$\phi_1 = \lambda_1 + \lambda_2 > 0 \tag{5.30}$$

which ensures the persistence of the state variable. This condition also implies that all the lag weights in Equation (5.21) are positive, which sets the lag structure at the same ground with the exponential Almon lag weighting scheme since the latter can only represent positive weights. As was briefly discussed, the lag weighting scheme in Equation (5.21) with positive λ s will take one of two forms: monotonically decreasing or single-hump shaped. Panel

B ($\lambda_1 = 0.6, \lambda_2 = 0.3$) and C ($\lambda_1 = 0.8, \lambda_2 = 0.6$) presents these examples and shows the exponential Almon lag weighting scheme does a very good job at replicating these structures.

Panel D ($\lambda_1 = 0.9, \lambda_2 = -0.8$) presents the last of its kind under Equation (5.30); one of the λ s is negative but the sum with the other one is positive. In this case, the lag weights will oscillate as in Panel D. Even in this case, the exponential Almon lag polynomial would do a pretty good job in reconstructing the state variable since it takes the averages of the oscillating weights and shocks on which these weights are put are supposed to be white noise.

6 Data and Empirical Results

We empirically investigate if the pricing information of empirically successful factors is associated with future market risk. We test if these factors are priced because they contain information about the future changes in the market return. The empirically successful factors we adopt here are the size, value, momentum, and liquidity. They are all tradable. The data for the first three factors are from Professor Kenneth French's website. The liquidity factor is the traded factor of Pastor and Stambaugh (2003). The factor data is from August 1962 to December 2014 except for the traded liquidity factor, which is available from January 1968 to December 2014. To match the universe of stocks used to generate the factors, we choose the return on the value-weighted portfolio of all the stocks on NYSE, AMEX and NASDAQ as the proxy for the market return.

We reconstruct the state variables from the factors and see if they predict the market return. To reconstruct the state variable, we need to assume a statistical process for it. First, we assume that the state variable follows AR(1). As shown in Section 5.2, the MIDAS regression specification can exactly reconstruct the state variables from the factors that are supposed to be innovations to the state variables. Second, we allow more general statistical process (*e.g.*, AR(2) or a more persistent process) for the state variable. The

flexible weighting scheme of the exponential Almon lag structure of the MIDAS regression would be a good approximation for the true weighting scheme as shown in Section 5.3. The latter case also allow us to deal with the autocorrelated factors.

For our empirical study, we use the maximum likelihood estimator (MLE) to estimate the parameters of the model, which requires the specification of the conditional variance. We add the GARCH(1,1) process for the market return volatility, and our empirical model is:

$$r_{m,t+1} = a_0 + a_1 X_t + \epsilon_{t+1} \quad (6.1)$$

$$= b_0 + b_1 \sum_{j=0}^J \frac{\exp(\theta_1 j + \dots + \theta_K j^K)}{S_e} f_{t-j}^X + \epsilon_{t+1} \quad (6.2)$$

$$h_{t+1} = \omega + \alpha \epsilon_t^2 + \beta h_t \quad (6.3)$$

where $\epsilon_t = \sqrt{h_t} \nu_t$ and ν_t is white noise. Then, it follows:

$$r_{m,t+1} | \Phi_t \sim N \left(b_0 + b_1 \sum_{j=0}^J \frac{\exp(\theta_1 j + \dots + \theta_K j^K)}{S_e} f_{t-j}^X, h_{t+1} \right) \quad (6.4)$$

where Φ_t is the information set available at time t , and the corresponding log likelihood function (LLF) would be:

$$LLF = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln(h_t) - \frac{1}{2} \sum_{t=1}^T \frac{\epsilon_t^2}{h_t} \quad (6.5)$$

We maximize the LLF to estimate the relevant parameters.

Table I provides various statistics for the market return proxy and other traded factors. For our sample period, the momentum strategy generates the largest profit followed by the one based on the market liquidity. The first order autocorrelations are estimated and examined because these factors are supposed to be innovations to the state variable and, if not, we need to take care of the serial correlation to reconstruct the state variable. Indeed,

some of the factors show significant serial correlation; the HML has large and statistically significant first order autocorrelation while the LIQ has second largest serial correlation with a quite small p -value.

The State Variable Follows an AR(1)

When the state variable follows an AR(1) process and the factors are innovations to the state variable, we can reconstruct the unknown state variable from the factors as in Equations (5.8) and (5.9). The empirical model specified in Equations (6.2) and (6.3) with $K = 1$ can exactly reconstruct the state variable. The models are fitted with the market return series from January 1970 to December 2014 and seven years of monthly lags ($J = 84$) of each of the factors mentioned except the liquidity one. Due to the data availability issue, the period February 1975 to December 2014 is used in the case of the traded liquidity factor. The choice of the length of the lags (*i.e.*, J) may seem arbitrary, but we believe our choice of $J = 84$ is enough to cover even a very persistent state process; for the state variable process with $\phi = 0.95$ in Equation (5.5), the weight on the last lag in Equation (5.6) is $0.95^{84} = 0.013$.

Table II shows the estimation results. The key parameter is b_1 . The statistical significance of b_1 implies that the reconstructed state variable X_t has predictive power over the market return, which in turn implies that the factor used to reconstruct the state variable is priced because it has information about the future changes in the market return. Table II shows that none of the b_1 s are statistically significant. In addition, none of the θ_1 s are statistically significant. For an intuitive understanding of how much of the variation of the market return can be explained by the reconstructed state variable, pseudo R^2 is also measured as follows:

$$\tilde{R}^2 = 1 - \frac{RSS}{TSS} \quad (6.6)$$

where $RSS = \sum_{t=1}^T (r_{m,t} - \bar{r}_m)^2$, $TSS = \sum_{t=1}^T \epsilon_t^2$, and $\bar{r}_m = 1/T \sum_{t=1}^T r_{m,t}$. \tilde{R}^2 is always very

low and less than 1%.

More General State Variables

The previous investigation with the AR(1) state variable does not show any statistically significant relation between the factors' pricing information and future market risk, and it seems worthwhile to examine it further, with a more general state variable process.

Given that the unknown state variable X_t follows an AR(2) process, the infinite order MA representation of state variable is shown in Equation (5.21). Assuming that the empirical factors are innovations to the state variable, we would like to reconstruct the state variable from the empirical factors and see if it predicts our choice of the aggregate measure, the market return, as specified in Equation (5.22). On the other hand, Section 5.3 and Figure 2 show that the exponential Almon lag structure in the MIDAS regression does a very good job at replicating the lag structure given in Equation (5.21). Hence, we do the corresponding empirical work with our empirical model in Equation (6.2) and (6.3) with $K = 2$. The flexible structure of the exponential Almon lag weighting scheme will also allow us to get a good approximate weighting structure even for more general state variable process.

Table III presents the empirical results. For a more general state variable, it turns out b_1 for SMB and LIQ are statistically significant with the standard MLE t -stats, but lose their statistical significance when the t -stats are computed with robust standard errors. Accordingly, the parameters (θ_1 and θ_2) for the weighting scheme are not statistically significant. The pseudo R^2 is still low with 1.9% and 0.7% for SMB and LIQ, respectively. Figure 3 shows the relation between the market return and the state variable reconstructed from the SMB. It seems the empirical finding confirms that the monthly market return is not very predictable and the innovation to any predictor would contain very noisy information about the future changes in the monthly market return.

The extension to a more general state variable case also allows us to deal with serially

correlated factors as in Equations (5.10a) and (5.10b). The infinite order MA representation of the state variable process in this case involves the innovation to the factor itself; see Equations (5.10b) and (5.24). The innovations (u_t) to the factors are obtained by the regression of the following specification:

$$f_{t+1} = \kappa_0 + \kappa_1 f_t + u_{t+1}$$

Table IV shows the various statistical properties of these innovation series. The correlation between the innovation and its original factor is very high, with over 99% for most of them, which implies other statistical properties such as standard deviation, skewness, and kurtosis are all similar to those from the original factors. The autocorrelation is also measured for each of the innovation series and we can see the serially correlated components are surely taken out.

Table V provides the empirical results with the factor innovations as specified in Equation (5.28), and confirms the previous outcome in Table III. b_1 s for SMB and LIQ are statistically significant with the standard MLE t -stat but lose the statistical significance when the robust standard errors are used to compute the t -stat. The parameters for the weighting function are not statistically significant either. Table I shows that HML is strongly and LIQ is somewhat autocorrelated. If the serial correlation has some effect in the estimation for Table II, we might see different results for HML and LIQ in Table V. The empirical evidence that LIQ is associated with the market return is strengthened to some extent with higher t -stat for b_1 and higher \tilde{R}^2 , but there is still no evidence that HML is related to future changes in market return.

Applications to Other Aggregate Measures and More

There are other aggregate measures that describe the investment opportunity set in the ICAPM. Empirical studies on the relationship between these other aggregate measures and

the traded factors are beyond the scope of this paper, but can be easily implemented in the same way we did here. In fact, Sohn (2009) runs an empirically similar type of test with the market volatility and provides strong and robust evidence that the SMB, WML and LIQ have information about the future changes in the market volatility.

As we discussed in Section 4, the sign of the predictive relation between the aggregate measure and state variable puts restrictions on the sign of the price of risk for the empirical factor. These sign relationships could be used to test the empirical factors further within the ICAPM framework. However, it should be noted that these sign-related restrictions become far more complicated when multiple aggregate measures are assumed in the version of the ICAPM adopted and there is a possibility that a state variable predicts multiple aggregate measures. In fact, for some sets of certain sign combinations across the aggregate measures, the restriction on the sign of the price of risk does not exist; it can be positive, negative or zero.

7 Concluding Comments

Harvey, Liu, and Zhu (2015) show that 240 factors are reported in the literature on the cross-section of equity returns. It is hard to believe that all 240 factors represent different systematic risks. To understand the nature of risk these pricing factors represent, it is necessary to check the theories developed in the field. The ICAPM is still a very popular choice for this purpose because it can be easily transformed into a linear factor pricing model and is very accommodating in the choice of state variables. However, the ICAPM is not open to any state variable and indeed has some restrictions on its choice. Hence, it is crucial to test these constraints on the state variables when we try to understand the nature of risks the empirical factors represent in the ICAPM framework.

Unfortunately, there are a number of empirical difficulties in testing and understanding the empirical factors in the ICAPM framework. We propose a new time-series test that can

overcome these difficulties and help us understand the nature of risk these empirical factors represent. Interestingly enough, as we discussed in the last part of Section 3, many empirical factors can represent the same systematic risk (*e.g.*, unexpected changes in the future market risk) and still command separate risk premiums. This also help us understand why we find so many empirically significant pricing factors.

Our empirical work in this paper is carried out with just one choice of aggregate measure: the market return. There are other prominent aggregate measures like the market volatility, the maximum Sharpe ratio, or the risk-free rate depending on the underlying assumptions of the version of the ICAPM. Our choice of aggregate measure is especially challenging in the sense that the expected component of the monthly market return is very small. Accordingly, we find little evidence that the risk premiums of the empirical factors are related to investors' concerns about the future changes in market risk.

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Table I: Summary Statistics for Trading-Strategy Based Factors

The table reports the summary statistics for market return and trading-strategy based factors. These are monthly returns to some portfolios from August 1962 to December 2014 except for the liquidity factor. The traded liquidity factor data from Pastor and Stambaugh (2003) are available from January 1968. The first-order autocorrelation (κ_1) was estimated by regressions using the following specification:

$$f_{t+1} = \kappa_0 + \kappa_1 f_t + u_{t+1}$$

The corresponding t -stats and R^2 are reported.

	r_m	SMB	HML	WML	LIQ
Mean	0.00924	0.00226	0.00378	0.00687	0.00433
STD	0.04442	0.03093	0.02841	0.04211	0.03517
Skewness	-0.53495	0.53599	-0.00443	-1.41821	0.43770
Kurtosis	5.07130	8.48605	5.61593	14.20293	5.63863
Autocorr	0.07280	0.04656	0.16050	0.05758	0.08044
t -stat	(1.83)	(1.17)	(4.07)	(1.44)	(1.91)
R^2	0.00530	0.00216	0.02576	0.00331	0.00647

Table II: Reconstruction of the AR(1) State Variables from Factors

The table presents the estimation results for the model specification in Equations (6.2) and (6.3) with $K = 1$. The model assumes the AR(1) process for the state variable to which the trading strategy based factors are the innovations as in Equations (5.8) and (5.9). The parameters are estimated with the MLE by maximizing the loglikelihood function in Equation (6.5). The models are fitted with market return series from January 1970 to December 2014 and seven years of monthly lags ($J = 84$) of each of the factors mentioned except for the liquidity one. Due to the data availability issue, the period from February 1975 to December 2014 is used in case of the traded liquidity factor. LLF is the value of the loglikelihood function, and \tilde{R}^2 is the pseudo coefficient of determination that shows how much of the variance of the market return can be explained by the variance of the reconstructed state variable from each of the factors. The numbers in parenthesis are the t -stats.

	b_0	b_1	ω	α	β	θ_1	LLF	\tilde{R}^2
SMB	0.00999 (5.81)	0.09154 (1.50)	0.00010 (1.90)	0.12209 (3.83)	0.83717 (23.21)	-15.30263 (-0.01)	918.7598	0.0041
HML	0.01069 (6.03)	-0.16252 (-1.39)	0.00010 (1.89)	0.12215 (3.82)	0.83673 (22.93)	-0.99298 (-0.74)	919.2664	0.0062
WML	0.01226 (5.13)	-0.36896 (-1.33)	0.00009 (1.85)	0.12131 (3.91)	0.84156 (24.59)	-0.08810 (-1.14)	919.0405	0.0070
LIQ	0.01087 (6.05)	-0.03773 (-0.68)	0.00007 (1.74)	0.11022 (3.60)	0.86152 (27.83)	-15.37705 (-0.01)	825.7719	0.0013

Table III: Reconstruction of More General State Variables from Factors

The table presents the estimation results for the model specification in Equations (6.2) and (6.3) with $K = 2$. The model assumes the AR(2) or more general process for the state variable to which the trading strategy based factors are the innovations as in Equations (5.27) and (5.28) with u replaced by the factors. The parameters are estimated with the MLE by maximizing the loglikelihood function in Equation (6.5). The models are fitted with market return series from January 1970 to December 2014 and seven years of monthly lags ($J = 84$) of each of the factors mentioned except for the liquidity one. Due to the data availability issue, the period from February 1975 to December 2014 is used in case of the traded liquidity factor. LLF is the value of the loglikelihood function and \tilde{R}^2 is the pseudo coefficient of determination that shows how much of the variance of the market return can be explained by the variance of the reconstructed state variable from each of the factors. The numbers in parenthesis and brackets are the t -stats from the standard errors and the robust standard errors (Huber sandwich estimator), respectively.

	b_0	b_1	ω	α	β	θ_1	θ_2	LLF	\tilde{R}^2
SMB	0.00850 (4.59) [3.45]	0.45578 (2.15) [1.16]	0.00010 (1.84) [1.56]	0.12603 (3.72) [2.91]	0.83396 (21.08) [14.74]	13.04126 (0.66) [0.30]	-0.14303 (-0.67) [-0.30]	923.6866	0.0190
HML	0.00921 (5.19)	0.17553 (1.71)	0.00009 (1.85)	0.11678 (3.85)	0.84733 (25.22)	38.45132 (0.88)	-0.65864 (-0.88)	919.3325	0.0069
WML	0.01237 (5.38)	-0.40208 (-1.62)	0.00009 (1.85)	0.12227 (3.94)	0.84150 (24.90)	0.04766 (0.18)	-0.00453 (-0.52)	919.2037	0.0071
LIQ	0.01013 (5.47) [4.45]	0.14821 (1.98) [1.76]	0.00007 (1.81) [1.85]	0.11545 (3.43) [2.94]	0.85722 (26.46) [25.82]	32.24338 (0.00) [0.16]	-10.47174 (-0.00) [-0.16]	827.6666	0.0069

Table IV: Summary Statistics for Innovations to Factors

The table reports the summary statistics for the innovations to the factors, u_t in Equations (5.10a) and (5.10b). These innovations are estimated using the following specification:

$$f_{t+1} = \kappa_0 + \kappa_1 f_t + u_{t+1}$$

The table presents the correlations between the innovations and the corresponding factors and other regular summary statistics for these innovation series. It also reports the first-order autocorrelations.

	u^{SMB}	u^{HML}	u^{WML}	u^{LIQ}
$Corr(f, u)$	0.99892	0.98703	0.99834	0.99676
STD	0.03092	0.02806	0.04207	0.03507
Skewness	0.50692	0.00705	-1.36184	0.42726
Kurtosis	8.75935	6.05861	13.77942	5.77388
Autocorr	-0.00142	-0.00227	0.00446	0.00804
t -stat	(-0.04)	(-0.06)	(0.11)	(0.19)
R^2	0.00000	0.00001	0.00002	0.00006

Table V: Reconstruction of More General State Variables from Factor Innovations

The table presents the estimation results for the model specifications in Equations (6.2) and (6.3) with $K = 2$. The model assumes the AR(2) or a more general process for the state variable to which the trading strategy based factors are the innovations as in Equations (5.27) and (5.28). In addition, it is assumed the factors are autocorrelated. The parameters are estimated with the MLE by maximizing the loglikelihood function in Equation (6.5). The models are fitted with market return series from January 1970 to December 2014 and seven years of monthly lags ($J = 84$) of each of the factor innovations except for the liquidity one. Due to the data availability issue, the period from February 1975 to December 2014 is used in case of innovations to the traded liquidity factor. LLF is the value of the loglikelihood function and \tilde{R}^2 is the pseudo coefficient of determination that shows how much of the variance of the market return can be explained by the variance of the reconstructed state variable from each of the factors. The numbers in parenthesis and brackets are the t -stats from the standard errors and the robust standard errors (Huber sandwich estimator), respectively.

	b_0	b_1	ω	α	β	θ_1	θ_2	LLF	\tilde{R}^2
u^{SMB}	0.00952 (5.60) [4.91]	0.47694 (2.15) [1.17]	0.00010 (1.85) [1.58]	0.12612 (3.73) [2.93]	0.83385 (21.14) [14.87]	13.01674 (0.67) [0.30]	-0.14260 (-0.68) [-0.31]	923.6916	0.0190
u^{HML}	0.00988 (5.74)	0.20362 (1.69)	0.00009 (1.86)	0.11673 (3.86)	0.84751 (25.30)	36.39428 (0.90)	-0.62030 (-0.90)	919.3552	0.0069
u^{WML}	0.00979 (5.74)	-0.12531 (-1.82)	0.00010 (1.89)	0.12128 (3.83)	0.83640 (22.70)	79.64523 (0.56)	-2.23483 (-0.55)	920.3791	0.0108
u^{LIQ}	0.01077 (6.08) [5.73]	0.15758 (2.03) [1.84]	0.00007 (1.76) [1.72]	0.11546 (3.63) [3.24]	0.85739 (27.05) [26.94]	28.23750 (0.01) [0.14]	-9.09779 (-0.01) [-0.14]	827.9041	0.0076

Figure 1: Exponential Almon Lag Weighting Scheme

This figure presents the flexibility of the exponential Almon lag weighting scheme, as defined by Equation (5.2) by showing various weighting types it can represent. The chosen length (J) of the lags for this example is 48 and $K = 2$. The chosen parameters are; (1) $\theta_1 = 0.005, \theta_2 = 0.001$, (2) $\theta_1 = 0.06, \theta_2 = -0.004$, (3) $\theta_1 = -0.14, \theta_2 = 0.003$, (4) $\theta_1 = -0.05, \theta_2 = -0.001$.

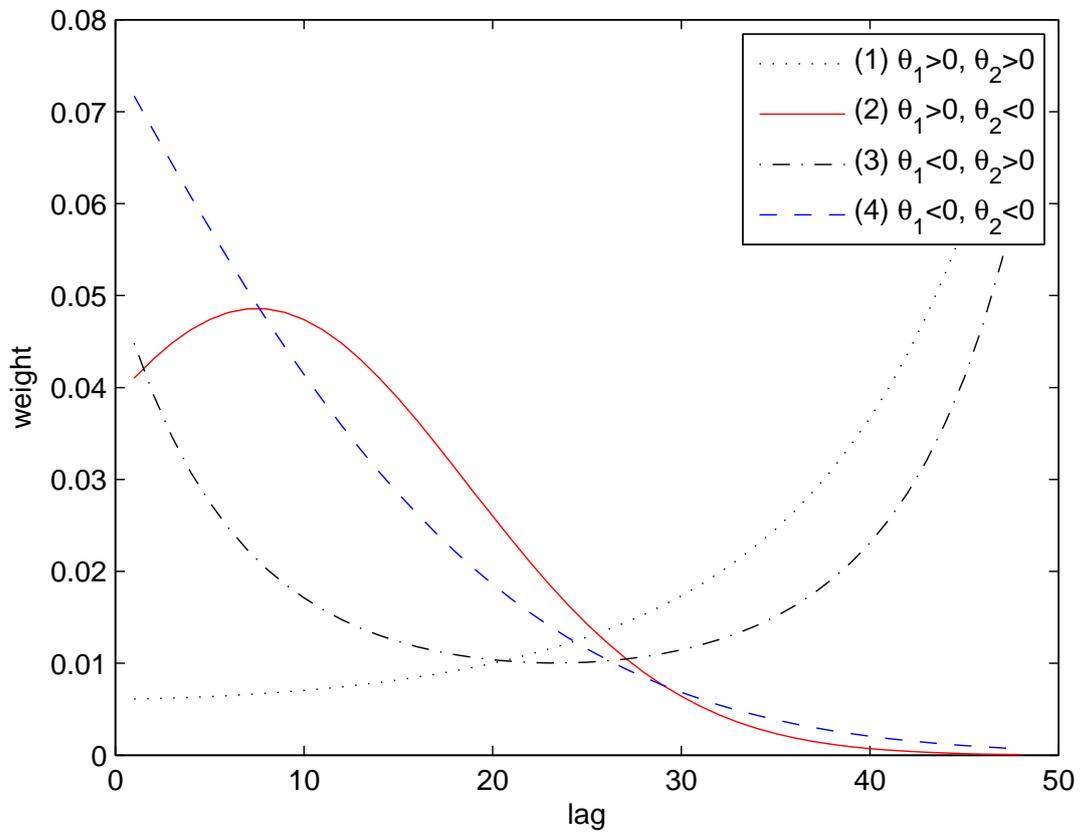


Figure 2: Lag Structure Approximation

This figure shows lag weighting schemes for the infinite order MA process represented by Equation (5.21) with various set of λ values: (A) $\lambda_1 = 0.95, \lambda_2 = 0.15$ (B) $\lambda_1 = 0.6, \lambda_2 = 0.3$ (C) $\lambda_1 = 0.8, \lambda_2 = 0.6$ (D) $\lambda_1 = 0.9, \lambda_2 = -0.8$. The weights are normalized by their sum such that the sum of all the weights is one as in Equation (5.27). The exponential Almon lag weighting scheme as in Equation (5.2) with $K = 2$ is used to match these lag structures as close as possible. The parameters were chosen to minimize the mean squared errors. The chosen length (J) of the lags for this example is 48.

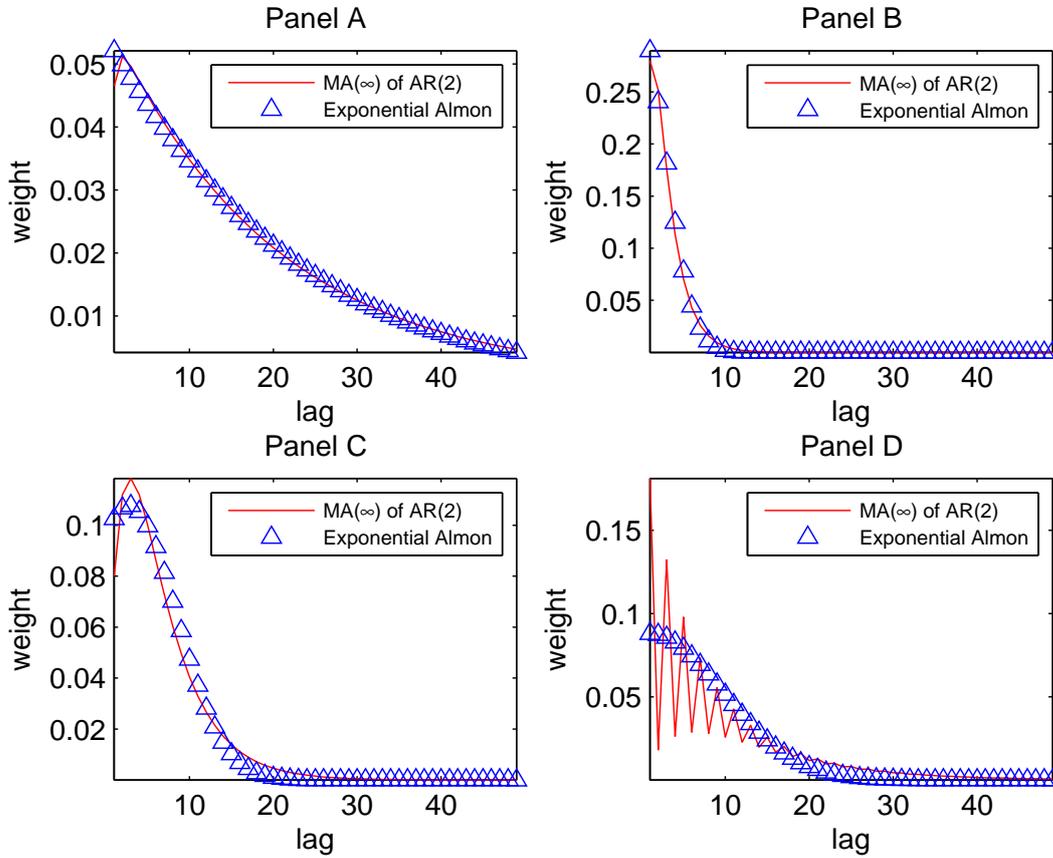


Figure 3: Market Return vs. the State Variable with SMB

This figure shows the market return and the state variable reconstructed with the SMB from Table III. The state variable is estimated with the model specification in Equations (6.2) and (6.3) with $K = 2$. The parameters are estimated with the MLE by maximizing the loglikelihood function in Equation (6.5). The market return series from January 1970 to December 2014 are fitted with seven years of monthly lags ($J = 84$) of the SMB factors.

